# ELEMENTS OF DECISION THEORY

(References: Stevenson W. J.: Operations management, Temesvári J.: A döntéselmélet alapjai (Foundations of Decision theory)) (Software: WinQSB (**Q**uantitative **S**ystem for **B**usiness) http://www.econ.unideb.hu/sites/download/WinQSB.zi

## 1. Introduction

Decision theory: web Google search= 10 million entries
Döntéselmélet: web Google keresés= 8 thousand entries

Some areas of decision theory:

- medical,
- law,
- **economics**,
- technical, engineering,
- othres.

Some characteristic problems in decision theory

Every day we have to make decisions. Sometimes we have to decide at once (e.g. what to eat for breakfast, where to have lunch today, which way to go to the university from our flat) at other occasions we have time to decide, moreover we must have justified, well considered decisions (e.g. decision about the productions of a factory: the quantity of various products, which are optimal in certain respect).

**1. Production problem**: a factory produces several products, and we have to decide how much to manufacture from each product such that e.g. the profit should be maximal, or the aim can be maximal profit with minimal use of energy (or labor) during the production process.

**2. Investment problem**: to choose a portfolio with maximal yield.
Constraint: financial, other points to consider risk factors, duration of the investment etc.

**3. Work scheduling**: e.g. a supermarket employs a certain number of workers. On each day, depending on the trade a certain number of workers have to work. We have to make a weekly schedule of the available workers such that the total weekly wage of the workers be minimal (wage for Saturday is higher than other days).

**4. Buying fighter planes**.
Points to consider: cost, max. speed, reliability etc.

**5. Tender evaluation**. An international bank wants to replace its computers with new ones. How to decide which offer to accept?

Points to consider: cost, quality of hardware, service conditions, guarantees,etc.

In each case the aim is one action: the best production plan, the highest return, optimal work schedule, finding the best fighter planes for the country, etc.

## Basic concepts

**Alternatives**: the different actions from which we can choose, their sets is called decision space.

Their characteristic properties:

- number of alternatives
- how can one characterize the alternative by numbers,
- independence of alternatives,
- probability of the alternatives

**Objectives (criterions, evaluation factors, aims)**: the directions to which we want to take our system (sometimes these cannot be reached, or cannot be expressed by numbers).

**Properties of objectives**:

- complete (all important objectives have to be present),
- should be suitable for scrutinizing,
- decomposable (the alternatives can be studied for each objective separately),
- omission of repetition of objectives,
- minimal (the number of objectives be the smallest possible),

**Decision makers**: the persons responsible for

- supplying the informations needed,
- determining the alternatives, choosing the best alternative(s),
- realizing the best alternative(s).

Behavior of decision makers: rational (wants to achieve optimum) or irrational.

The decision makers see a part of the problems objectively (in particular those which can be measured, expressed by numbers, calculated values) in other parts of the problems some problems the decision makers may have preferences (subjective standpoint).

**Behavior science**: for the decision makers the principle of **restricted rationality** is valid

**The decision process:**

- formation of a decision situation (there is a need for solving conflicts),
- phrasing of the decision problem,
- formalizing of the decision problem (in terms of mathematics),
- choosing the method of solution,
- solution (choosing the optimal action),
- adapting, evaluation, scrutinizing: was the decision correct or we have to begin the decision process again.

## 2. Buying fighter planes

Viewpoints of the experts:

- $S_1$= max. speed km /h,
- $S_2$= useful area in the plane in $m^2$,
- $S_3$= loadability in kg,
- $S_4$= cost in million dollars,
- $S_5$= reliability,
- $S_6$= ability to maneouver.

$S_5, S_6$ are evaluated in the following way:
vl=very low, l=low, a=average, g=good, vg=very good.

The table of offers (alternatives:

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $S_1$ | 1000 | 1250 | 900 | 1100 |
| $S_2$ | 150 | 270 | 200 | 180 |
| $S_3$ | 20000 | 18000 | 21000 | 20000 |
| $S_4$ | 5, 5 | 6, 5 | 4, 5 | 5, 0 |
| $S_5$ | a | l | g | g |
| $S_6$ | vg | a | g | a |

## Evaluation of quantitative points of view

| | |
|---|---|
| vl=very low | =1 points |
| l=low | =3 points |
| a=average | =5 points |
| g=good | =7 points |
| vg=very good | =9 points |

## Making data independent of units

Ideal values are given by experts, or we have prior information on them.

The data of our table (matrix) are denoted by: $x_{ij}$ the number in the $i$th line, and the $j$th column (elements of a $6 \times 4$ type matrix)

Ideal value in the $i$th line: $\max_j x_{ij}$, (where the maximum is taken for all index $j$) **if the largest value is the ideal one**, and $\min_j x_{ij}$, **if the smallest value is the ideal one.**

The transformed value

$$v_{ij} = \frac{x_{ij}}{\max_j x_{ij}},$$

if the largest value is the ideal one, and

$$v_{ij} = \frac{\min_j x_{ij}}{x_{ij}},$$

if the smallest value is the ideal one.

Therefore, if

$i = 1$ then $\max_j x_{1j} = 1250$, $v_{1j} = \frac{x_{1j}}{1250}$ $(j = 1, 2, 3, 4)$

$i = 2$ then $\max_j x_{2j} = 270$, $v_{2j} = \frac{x_{2j}}{270}$ $(j = 1, 2, 3, 4)$

$i = 3$ then $\max_j x_{3j} = 21000$, $v_{3j} = \frac{x_{3j}}{21000}$ $(j = 1, 2, 3, 4)$

$i = 4$ then $\min_j x_{4j} = 4,5, !!!$ $v_{4j} = \frac{4,5}{x_{4j}}$ $(j = 1, 2, 3, 4)$

$i = 5$then $\max_j x_{5j} = 7,$ $v_{5j} = \frac{x_{5j}}{7}$ $(j = 1, 2, 3, 4)$

$i = 6$ then $\max_j x_{6j} = 9,$ $v_{6j} = \frac{x_{6j}}{9}$ $(j = 1, 2, 3, 4)$

Our new table:

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $S_1$ | $0,80$ | $1$ | $\mathbf{0,72}$ | $0,88$ |
| $S_2$ | $\mathbf{0,56}$ | $1$ | $0,74$ | $0,67$ |
| $S_3$ | $0,95$ | $0,86$ | $1$ | $0,95$ |
| $S_4$ | $0,82$ | $0,64$ | $1$ | $0,90$ |
| $S_5$ | $0,71$ | $\mathbf{0,43}$ | $1$ | $0,71$ |
| $S_6$ | $1$ | $0,56$ | $0,78$ | $\mathbf{0,56}$ |

The minimal elements in each column are written in boldface letter. Each element of the matrix is between 0 and 1 and every line contains a 1 (namely the best offer(s))

In the **columns of the alternatives (offers) the best value is 1, and the smallest value is the worst.**

## Decision methods: narrowing the alternatives

**(a) Method of satisfaction**: there is a level for each alternative which is just satisfactory, below this level we cannot accept the alternative(s). For example at University mark less than 2 is failure!

**(b) Disjunctive method:** *gives priority to excellence* (e.g. sport, science, art). If an alternative is better than a given level, then it is acceptable.

**(c) Dominated alternatives.** *An alternative is dominated by another one* if it is worse in every point of view than the other. Rational decision maker does not choose dominated alternative.

## Decision based on the order of importance: lexicographic method

Order the alternatives according to some point of view, and decide accordingly. For example if price is the most important (poor country cannot afford expensive planes) the we choose the cheepest offer.

## Decision methods: optimistic, pessimistic, Hurwicz index

In each column of the table we choose the minimal (worst) and the maximal (best) value, these are printed by boldface and italic numbers respectively:

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $S_1$ | $0,80$ | *1* | $\mathbf{0,72}$ | $0,88$ |
| $S_2$ | $\mathbf{0,56}$ | *1* | $0,74$ | $0,67$ |
| $S_3$ | $0,95$ | $0,86$ | *1* | *0,95* |
| $S_4$ | $0,82$ | $0,64$ | *1* | $0,90$ |
| $S_5$ | $0,71$ | $\mathbf{0,43}$ | *1* | $0,71$ |
| $S_6$ | *1* | $0,56$ | $0,78$ | $\mathbf{0,56}$ |

Denote thes minimal and maximal values by $m_j$ and $M_j$ respectivelt:

$$m_j := \min_i v_{ij}, \qquad M_j := \max_i v_{ij}.$$

**The pessimistic decision maker** first choose the worst value in each column and out of these finds the best value and takes the alternative corresponding to it (avoid the worst), this method is called the **maximin** method. In our case

$$\max_j m_j = \max_j \left( \min_i v_{ij} \right) = 0,72$$

the decision is $A_3$.

**The optimistic decision maker** first choose the best value in each column and out of these finds again the best value and takes the alternative corresponding to it.this method is called the **maximax** method. In our case

$$\max_j M_j = \max_j \left( \max_i v_{ij} \right) = 1$$

the decisions are $A_1, A_2, A_3$ which are equivalent.

We can introduce the **Hurwicz optimistic coefficient (index)** $\alpha \in [0,1]$, then first we calculate the quantities

$$H_j(\alpha) = \alpha M_j + (1-\alpha) m_j \qquad (j = 1, 2, 3, 4)$$

then find their maximum:

$$\max_j H_j = H_{j_0}$$

if maximum is attained at $j_0$ the the decision is the alternative $A_{j_0}$.
$\alpha = 0$ means pessimistic decision maker, $\alpha = 1$ is the optimistic one.

## 3. Decision in case of uncertainty: extension of a business

In the previous example the offers do not depend on random events, we had a **deterministic model**.
In a **stochastic model** our decision is influenced by random events.

Our business can be extended in 3 ways:
$A_1$ = new branch,
$A_2$ = new service,
$A_3$ = new product.

Our decision depends on the demand of next year, which should be estimated.

| demand of next year | estimated subjective probabilities |
|---|---|
| $S_1$ very good | $P(S_1) = 0,4$ |
| $S_2$ good | $P(S_2) = 0,3$ |
| $S_3$ medium | $P(S_3) = 0,2$ |
| $S_4$ weak | $P(S_4) = 0,1$ |

$$\sum_{i=1}^{4} P(S_i) = 1$$

The payoff table of the alternatives depends on the demands in the following way (given in million Ft) together with the probabilities is given in the next table:

| | $A_1$ | $A_2$ | $A_3$ | $P$ |
|---|---|---|---|---|
| $S_1$ | 20 | 26 | 10 | 0,4 |
| $S_2$ | 12 | 10 | 8 | 0,3 |
| $S_3$ | 8 | 4 | 7 | 0,2 |
| $S_4$ | 4 | $-4$ | 5 | 0,1 |

The elements of the matrix are denoted by $v_{ij} = v(S_i, A_j)$.

**Decision possibilities.**

**1. We neglect the probabilities** and decide on the basis of the payoffs.(giving equal chance to $S_1, S_2, S_3, S_4$).

**(a) Pessimistic, optimistic and Hurwicz index** see before.
**(b) Minimax regret decision** Regret is the payoff which is not realized.

If e.g.$S_1$ happens but we did not choose (the best payoff) $A_2$ but $A_1$ or $A_3$ then payoff not realized is

| $A_1$ | $A_2$ | $A_3$ |
|---|---|---|
| 6 | 0 | 16 |

Each elements of the line are deducted from the maximal element, in this way we get the regret table:

| | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| $S_1$ | 6 | 0 | 16 |
| $S_2$ | 0 | 2 | 4 |
| $S_3$ | 0 | 4 | 1 |
| $S_4$ | 1 | 9 | 0 |
| column maximum | 6 | 9 | 16 |

The minimum of these column maximums is =6, the decision is $A_1$.

**2. We take the probabilities into consideration.**

**(a) Decision based on (maximal) expected payoff**

The expected values of the payoffs for each alternative are:

$$\begin{aligned} E(A_1) &= 20 \cdot 0,4 + 12 \cdot 0,3 + 8 \cdot 0,2 + 4 \cdot 0,1 &&= 13,6 \\ E(A_2) &= &&= 13,8 \\ E(A_3) &= &&= 8,3 \end{aligned}$$

Decision: $A_2$.

**(b) Equal likelihood method.** Decision based on maximal expected value taking equal probabilities:

$$\begin{aligned} E(A_1) &= 20 \cdot 0,25 + 12 \cdot 0,25 + 8 \cdot 0,25 + 4 \cdot 0,25 &&= 11 \\ E(A_2) &= &&= 9 \\ E(A_3) &= &&= 7,5 \end{aligned}$$

Decision: $A_1$.

**(c) Decision based on (minimal) expected regret.**

The expected values of the regrets:

$$\begin{aligned}\tilde{E}(A_1) &= 6\cdot 0,4 + 0\cdot 0,3 + 0\cdot 0,2 + 1\cdot 0,1 &= 2,5\\ \tilde{E}(A_2) &= &= 2,3\\ \tilde{E}(A_3) &= &= 7,8\end{aligned}$$

Decision $A_2$.

**Expected value of perfect information.**

Suppose that in some way we can influence the state of events $S_i$ (e.g. we postpone the decision until the economy is booming and the demand is very good) and in each year we do know which state will be realized.

The probabilities of the events will remain the same, but repeating the expansion of our business through 100 years (of course in theory only) we make in each year the best decision.Out of 100 years

as $P(S_1) = 0,4$ the event $S_1$ is realized 40 times, we choose $A_2$ with payoff 26,
as $P(S_2) = 0,3$ the event $S_2$ is realized 30 times, we choose $A_2$ with payoff 12,
as $P(S_3) = 0,2$ the event $S_3$ is realized 20 times, we choose $A_1$ with payoff 8,
as $P(S_4) = 0,1$ the event $S_4$ is realized 10 times, we choose $A_3$ with payoff 5,
the average payoff for one year is

$$26\cdot 0,4 + 12\cdot 0,3 + 8\cdot 0,2 + 5\cdot 0,1 = 16,1$$

subtracting the expected value of the payoff we get **the expected value (payoff) of perfect information**:

$$16,1 - 13,8 = 2,3.$$

We can solve this problem by the Decision Analysis module of the software WinQSB.

Data entry to the Payoff table analysis problem:

| Decision \ State | S1(n. jó) | S2(jó) | S3(közepes) | S4(gyenge) |
|---|---|---|---|---|
| Prior Probability | 0.4 | 0.3 | 0.2 | 0.1 |
| A1(új fiók.) | 20 | 12 | 8 | 4 |
| A2(új szolg.) | 26 | 10 | 4 | -4 |
| A3(új term.) | 10 | 8 | 7 | 5 |

The table of solution:

| | | | |
|---|---|---|---|
| 02-28-2010 | Best | Decision | |
| Criterion | Decision | Value | |
| Maximin | A3(új term.) | 5 | |
| Maximax | A2(új szolg.) | 26 | |
| Hurwicz (p=0,8) | A2(új szolg.) | 20 | |
| Minimax Regret | A1(új fiók.) | 6 | |
| Expected Value | A2(új szolg.) | 13,80 | |
| Equal Likelihood | A1(új fiók.) | 11 | |
| Expected Regret | A2(új szolg.) | 2,30 | |
| | | | |
| Expected Value | without any | Information = | 13,80 |
| Expected Value | with Perfect | Information = | 16,10 |
| Expected Value | of Perfect | Information = | 2,30 |

# 3. Decision trees

This is a graphical decision method.

A company considers developing two new products.

The **first alternative** $A_1$ is a smoke and fire detector, the estimated development costs is 100000Ft, in case of success the expected increase of the profit is 1000000Ft and the probability of success is 0,5.

The **second alternative** $A_2$ is a motion detector, whose estimated development cost is 10000Ft, in case of success the expected increase of the profit is 400000Ft and the probability of success is 0,8.

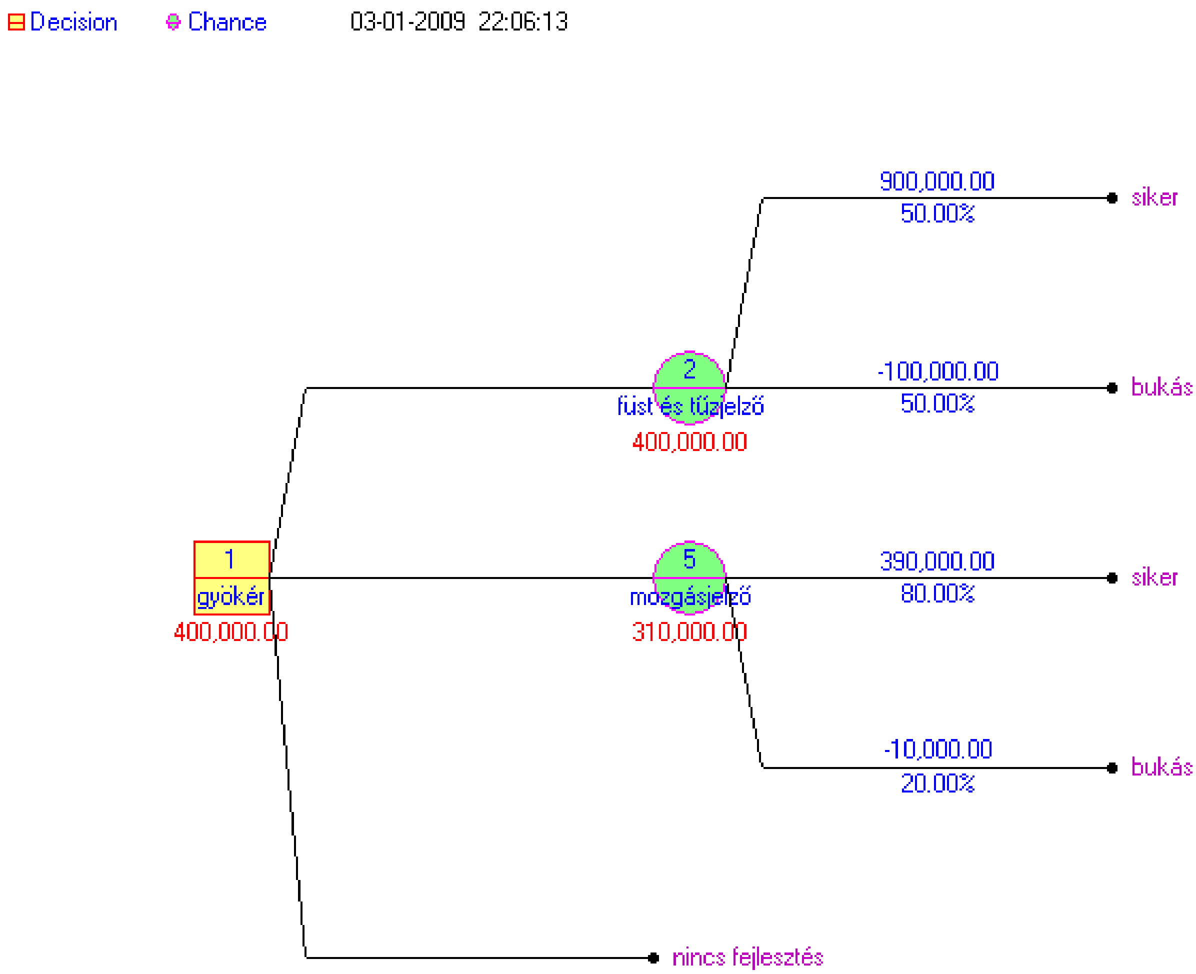
Of course there is a **third alternative** $A_3$ doing nothing (no new product).

In a decision tree we have three kind of nodes:

(1) decision node (denoted by a square)
(2) chance node, from which branches start with probabilities (denoted by a circle)
(3) endpoint (denoted by a black dot or triangle)

The starting node is called root. Starting from the root we draw branches to the right which run into a circle or square. If a branch starts at a circle then we write on it the corresponding probability, and continue until we reach the endpoint. The we calculate the expected values which we write under the chance nodes (circles). Going to the left we write under the decision nodes the smaller expected value.

The decision tree of the above problem is shown below.




We put the following data in the decision analysis program:

| Node/Event Number | Node Name or Description | Node Type ( D or C) | Immediate Following Node (numbers separated by ',') | Node Payoff (+ profit, - cost) | Probability (if available) |
|---|---|---|---|---|---|
| 1 | gyökér | d | 2,5,8 | | |
| 2 | füst és tűzjelző | c | 3,4 | | |
| 3 | siker | | | 900000 | 0.5 |
| 4 | bukás | | | -100000 | 0.5 |
| 5 | mozgásjelző | c | 6,7 | | |
| 6 | siker | | | 390000 | 0.8 |
| 7 | bukás | | | -10000 | 0.2 |
| 8 | nincs fejlesztés | | | 0 | |

**Method of solution**: first we draw the decision tree by hand, number the nodes, decide which is decision node which is chance node and also write in the tree the probabilities and the payoffs(profits). Next we open the Decision Analysis module of **WinQSB** then after clicking to **File/ New Problem /Decision Tree Analysis** a window opens to where we write

the name of the problem

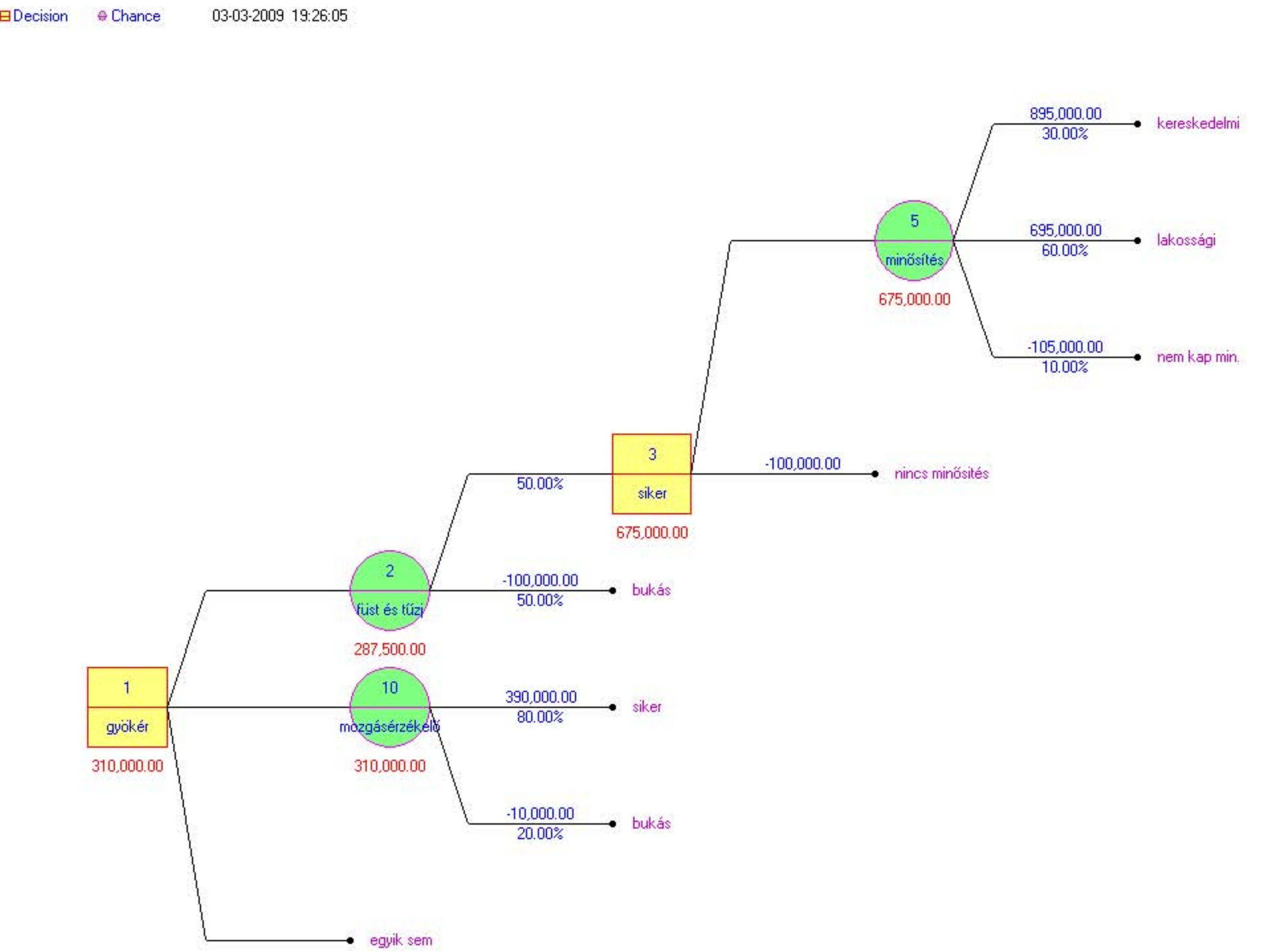
and give the number of nodes **OK**.

Again a window opens to which we write (using the hand drawn decision tree) the names of the nodes, the branchings, the types of the nodes, the payoffs and

Next click **Solve and Analyse, Draw Decision Tree** which opens again a window where we can modify the size of the tree, the nodes the data the program has to calculate. Click **OK** then the tree will be drawn which we can make nicer modifying the display data.

Modification of the previous problem. It turned out that the smoke and fire detector can be sold only after a quality control. The cost of this is 5000Ft. After the control the product can obtain three different quality grade: commercial quality, public quality and not qualified. The probability of obtaining commercial grade is 0,3 and in that case the net income from this product is 1000000Ft, while public grade has probability 0,6 and in this case the net income from this product is only 800000Ft. Probability of no qualification is 1-0,3-0,6=0,1.

The data of the modified problem and its decision tree:

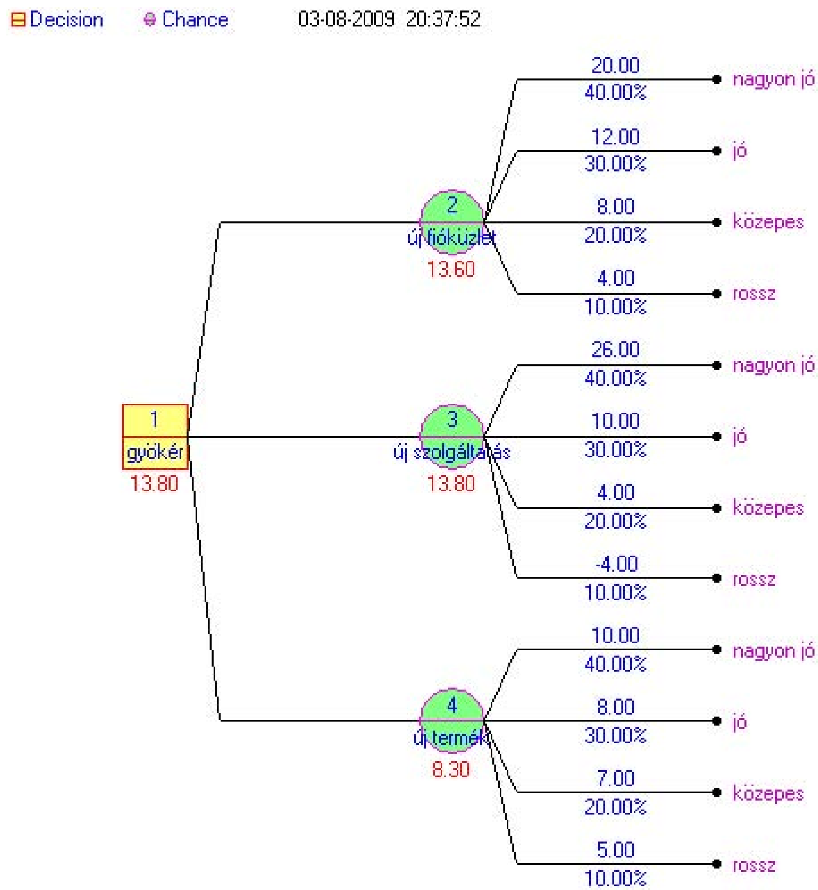



The d

| | | | | | |
|---|---|---|---|---|---|
| DA | uujterm | DT | 13 | | |
| Node/Event Number | Node Name or Description | Node Type (enter D or C) | Immediate Following Node (numbers separated by ',') | Node Payoff (+ profit, - cost) | Probability (if available) |
| 1 | gyökér | d | 2,10,13 | | |
| 2 | füst és tűzj | c | 3,4 | | |
| 3 | siker | d | 5,6 | | 0.5 |
| 4 | bukás | | | -100000 | 0.5 |
| 5 | minősítés | c | 7,8,9 | | |
| 6 | nincs minősités | | | -100000 | |
| 7 | kereskedelmi | | | 895000 | 0.3 |
| 8 | lakossági | | | 695000 | 0.6 |
| 9 | nem kap min. | | | -105000 | 0.1 |
| 10 | mozgásérzékelő | c | 11,12 | | |
| 11 | siker | | | 390000 | 0.8 |
| 12 | bukás | | | -10000 | 0.2 |
| 13 | egyik sem | | | 0 | |

The decision tree of the business extension (dealt with in 3.1) is seen below:

| Node/Event Number | Node Name or Description | Node Type (enter D or C) | Immediate Following Node (numbers separated by ',') | Node Payoff (+ profit, - cost) | Probability (if available) |
|---|---|---|---|---|---|
| 1 | gyökér | d | 2,3,4 | | |
| 2 | új fióküzlet | c | 5,6,7,8 | | |
| 3 | új szolgáltatás | c | 9,10,11,12 | | |
| 4 | új termék | c | 13,14,15,16 | | |
| 5 | nagyon jó | | | 20 | .4 |
| 6 | jó | | | 12 | .3 |
| 7 | közepes | | | 8 | .2 |
| 8 | rossz | | | 4 | .1 |
| 9 | nagyon jó | | | 26 | .4 |
| 10 | jó | | | 10 | .3 |
| 11 | közepes | | | 4 | .2 |
| 12 | rossz | | | -4 | .1 |
| 13 | nagyon jó | | | 10 | .4 |
| 14 | jó | | | 8 | .3 |
| 15 | közepes | | | 7 | .2 |
| 16 | rossz | | | 5 | .1 |


Decision
Chance
03-08-2009 20:37:52
1
gyökér
13.80
2
új fióküzlet
13.60
3
új szolgáltatás
13.80
4
új termék
8.30
20.00
40.00%
nagyon jó
12.00
30.00%
jó
8.00
20.00%
közepes
4.00
10.00%
rossz
26.00
40.00%
nagyon jó
10.00
30.00%
jó
4.00
20.00%
közepes
-4.00
10.00%
rossz
10.00
40.00%
nagyon jó
8.00
30.00%
jó
7.00
20.00%
közepes
5.00
10.00%
rossz