Discrete time interest rate models

slides for the course
"Financial Mathematics III (Interest rate theory"),
University of Ljubljana, 2018-19/II,
part II

József Gáll
University of Debrecen, Faculty of Economics

February – May 2019, Ljubljana

Basic notations, assumptions

- We shall consider a time interval $[0, T^*]$, where T^* is an integer, denoting the T^*th trading time point in the market.
- The trading times will be thus simply $0, 1, 2, 3, \ldots, T^* - 1, T^*$. Let $\mathcal{T}$ denote the set of trading times, i.e. define $\mathcal{T} = \{0, 1, 2, \ldots, T^*\}$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- $\mathbb{F} = \{\mathcal{F}_k\}_{k=0}^{T^*}$ is a filtration on it.
- The price of a T-bond at time t will be denoted by $P(t, T)$, assumed to be $\mathcal{F}_t$-measurable, where we also assume $P(T, T) = 1$, for all $T \in [0, T^*]$.
- The risk free spot rate corresponding to the time interval $[t, t+1)$ will be $r(t)$. Clearly, $r(t)$ is assumed to be $\mathcal{F}_t$-measurable, $t \in \mathcal{T}$.

Spot rate, bank account

- The bank account (or the risk-free asset) shall be an asset paying the risk-free return, hence it's price process shall be

$$B(0) = 1, \quad \text{and} \quad B(t) = e^{\sum_{s=0}^{t-1} r(s)}, \quad t \in \mathcal{T}.$$

Note that B is predictable w.r.t. the filtration $\mathbb{F}$.

- We also have for the spot rate in any model

$$P(t, t+1) = e^{-r(t)} \text{ and hence } r(t) = -\ln P(t, t+1),$$

$t \in \mathcal{T}$, $t < T^*$.

Forward rates

- $F(t, T_1, T_2)$ is the forward rate at time t corresponding to the time interval $[T_1, T_2]$ according to continuous compounding convention, where $0 \leq t \leq T_1 < T_2 \leq T^*$, $t, T_1, T_2 \in \mathcal{T}$.
- Note that it is enough to work only with forward rates corresponding to one-step intervals, so with rates of the form $F(t, T, T+1)$, since they determine the rest of the forward rates.
- We will have for the forward rates in any model

$$P(t, T+1) = P(t, T)\mathrm{e}^{-F(t,T,T+1)}$$

and hence

$$F(t, T, T+1) = \ln \frac{P(t, T)}{P(t, T+1)}, \quad t \leq T < T^*, \quad t, T \in \mathcal{T}.$$

Forward rates (cont.)

- More generally,

$$P(t,T) = \mathrm{e}^{-\sum_{s=t}^{T-1} F(t,s,s+1)}, \quad t \leq T < T^*, \;\; t, T \in \mathcal{T}.$$

- In particular, $r(t) = F(t,t,t+1)$, $t < T^*$, $t \in \mathcal{T}$.
- Note that the initial values $P(0,T)$, $T \in \mathcal{T}$, are given in the market, and hence the initial forward rates $F(0,T,T+1)$ and the spot rate $r(0) = F(0,0,1)$ are also given (known).

A deterministic market

Before turning to stochastic models, we consider the case of non-random interest rates and bond prices.
If the market excludes arbitrage then we have

$$r(T) = F(0, T, T + 1) = F(t, T, T + 1),$$

$0 \leq t \leq T < T^*$, $t, T \in \mathcal{T}$, since otherwise we would have two different deterministic (risk-free) rates over the interval $[T, T + 1]$.
Hence, given the initial bond prices $P(0, T)$, $T \in \mathcal{T}$, we have the initial forward rate, which determine the spot rates and the other forward rates (see the above equation).
In particular, for $0 \leq t \leq T \leq T^*$, $t, T \in \mathcal{T}$, we have
$P(t, T) = \exp(-\sum_{s=t}^{T-1} F(0, s, s + 1)) = P(0, T)/P(0, t)$.

Arbitrage free bond prices

Definition. We call the set of bond prices $P(t,T)$, $0 \leq t \leq T \leq T^*$ an arbitrage-free family of bond prices relative to the risk free rates $r(t)$ if

- there exists a prob. measure $\mathbb{P}^*$ such that it is equivalent to $\mathbb{P}$ and
- the discounted bond price processes

$$Z(t,T) = \frac{P(t,T)}{B(t)}, \quad 0 \leq t \leq T,$$

form $\mathbb{P}^*$-martingales for all maturity T.

Note that $\mathbb{P}^*$ is an equivalent martingale measure (EMM).

Bond prices under the EMM

In an arbitrage-free family of bond prices we have

$$
\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{P}^*}\left(\frac{B(t)}{B(T)}\,\Big|\,\mathcal{F}_t\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\sum_{s=t}^{T-1} r(s)}\,\Big|\,\mathcal{F}_t\right) \\
&= P(t,t+1)\mathbb{E}_{\mathbb{P}^*}\left(P(t+1,T)\,|\,\mathcal{F}_t\right).
\end{aligned}
$$

Trading strategies with bonds

Definition. A portfolio strategy with (finitely many) bonds is a sequence $\pi_n = (\beta_0^{(n)}, \beta_1^{(n)}, \ldots, \beta_M^{(n)})$, $0 \leq n \leq N$, for some integers $N > 0$ and $M \geq 0$ with $N + M \leq T^*$, where the random variable $\beta_i^{(n)}$ denotes the number of $(n+i)$-bonds such that it is assumed to be measurable w.r.t. $\mathcal{F}_{n-1}$.

The value of the strategy at time n is

$$X_n^\pi = \sum_{i=0}^{M} \beta_i^{(n)} P(n, n+i).$$

Self-financing strategies

Definition. A strategy π is said to be self-financing if

$$X^{\pi}_{n-1} = \sum_{i=0}^{M} \beta_i^{(n)} P(n-1, n+i).$$

Remark. The self-financing property is equivalent with

$$\sum_{i=0}^{M-1} \left(\beta_i^{(n)} - \beta_{i+1}^{(n-1)} \right) P(n-1, n+i)$$

$$+ \beta_M^{(n)} P(n-1, n+M) - \beta_0^{(n-1)} P(n-1, n-1) = 0.$$

Fundamental theorem of asset pricing

Theorem.

- In an arbitrage-free family of bond prices for a self-financing strategy π the discounted value process $\frac{X_t^\pi}{B(t)}$ is a $\mathbb{P}^*$-martingale.
- Hence one cannot make arbitrage strategy with bonds (i.e. one implication of the fundamental theorem of asset prices is valid.)

Binary probability space

- We assume that there is a 'binary randomness' behind the model, that is $|\Omega| = 2^{T^*}$. For instance, one might take $\Omega = \{ (x_1, x_2, \ldots, x_{T^*}) \mid x_i \in \{a_i, b_i\},\ i = 1, 2, \ldots, T^* \}$, where a_i refers to a 'downward jump', whereas b_i to an 'upward jump' at time (or step) i.
- Note that 'upward' will only mean in the models that the price change corresponding to this (b_i) is larger than that of the 'downward' one (a_i). (But both may cause decrease or increase.)

Filtration

- Define $\mathcal{F}_{T^*} := 2^{\Omega}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- For an integer i with $0 \le i \le T^*$, and values $y_j \in \{a_j, b_j\}$, $j = 1, 2, \ldots, t$, define

$$(y_1, y_2, \ldots, y_i, \cdot)$$
$$= \{\, (x_1, x_2, \ldots, x_{T^*}) \in \Omega \mid x_j = y_j,\ j = 1, 2, \ldots, i \,\}.$$

Then one can naturally define a filtration by defining $\mathcal{F}_t$ to be the σ-algebra generated by all events $(y_1, y_2, \ldots, y_i, \cdot)$ with $1 \le i \le t$, i.e. by all events which are determined by the first t steps of the market.

Assumptions in the Ho-Lee model

Given a binary filtered probability space described above for $t \in \mathcal{T}$, $t < T^*$ let

$$P(t+1, T) = \begin{cases} u(t, T-t)e^{r(t)}P(t, T) & \text{if at time } t+1 \\ & \text{the market goes up,} \\ d(t, T-t)e^{r(t)}P(t, T) & \text{if at time } t+1 \\ & \text{the market goes down,} \end{cases}$$

where $u(t, T-t)$, $d(t, T-t)$ are real numbers with $u(t, T-t) \geq d(t, T-t) > 0$. (Note that one could consider, more generally, random coefficients $u(t, \cdot)$ and $d(t, \cdot)$ being adapted.)

Assumptions in the Ho-Lee model (cont.)

- Clearly, $u(t, T-t)e^{r(t)}P(t, T) = u(t, T-t)\frac{P(t,T)}{P(t,t+1)}$, and $d(t, T-t)e^{r(t)}P(t, T) = d(t, T-t)\frac{P(t,T)}{P(t,t+1)}$.
- Note that this way $P(t, T)$ is defined to be $\mathcal{F}_t$ measurable.
- We only consider the case where there are constants $u(s)$, $d(s)$, $s \in \mathcal{T}$, $s \geq 2$ such that $u(s) = u(t,s)$ and $d(s) = d(t,s)$ for all possible $t \in \mathcal{T}$, $t < T^*$, hence the values of the coefficients are homogeneous in time (t).
- We set $u(1) = d(1) = 1$.

No-arbitrage in the Ho-Lee model

Theorem. Consider the Ho-Lee model described above and assume that it is free of arbitrage.

- Then we have $d(s) < 1 < u(s)$ for all $s \in \mathcal{T}$, $s \geq 2$.
- Let us write

$$q(s) := \frac{1 - d(s)}{u(s) - d(s)}, \quad s \in \mathcal{T}, s \geq 2.$$

  Then there exists a $q \in (0, 1)$ such that $q = q(s)$ for all $s \in \mathcal{T}$, $s \geq 2$.

No-arbitrage in the Ho-Lee model (cont.)

Theorem. (cont.)

- Furthermore, define the probability measure $\mathbb{P}^*$ as follows. For $\omega = (x_1, x_2, \ldots, x_{T^*}) \in \Omega$ let

$$\mathbb{P}^*(\omega) := q^k (1-q)^{T^*-k},$$

where k is an integer $(0 \leq k \leq T^*)$ denoting the number of upward jumps in ω (i.e. the number of cases where $x_i = b_i$, $i \in \mathcal{T}$).
Then $\mathbb{P}^*$ is an EMM in the market.

No-arbitrage in the Ho-Lee model (cont.)

Remark.

- Given an EMM in the Ho-Lee market, the market clearly excludes arbitrage, hence the measure will be of the form given in the previous theorem.
- If we are given $\mathbb{P}^*$ then $u(s) = \frac{1-d(s)(1-q)}{q}$ for all $s \in \mathcal{T}$, $s \geq 2$.

Binomial Ho-Lee model

Suppose now that we take the special case where the bond trees are all binomial trees. This means (like in any binomial market model) that only the number of upward jumps determine the value of the assets after a certain number of steps in the market, but the times of the upward jumps do not matter.

For the second step therefore we must have the same value for $P(2, T)$, in case where the market first goes up and then down, or in case where the market first goes down and then up. More formally, $P(2, T)(\omega) = P(2, T)(\omega')$ for all $\omega \in (a_1, b_2, \cdot)$ and $\omega' \in (b_1, a_2, \cdot)$.

Binomial Ho-Lee model (cont.)

Corollary. In a binomial arbitrage-free Ho-Lee model we have

$$\frac{d(s)}{u(s)} = k^{s-1}, \quad s \in \mathcal{T},\ s \geq 2,$$

where $k := \frac{d(2)}{u(2)}$.

Forward rate dynamics in binomial Ho-Lee market

Theorem. In the arbitrage-free binomial Ho-Lee model we have the following dynamics for the forward rates:

$$F(t, T-1, T) = \ln \frac{P(t, T-1)}{P(t, T)}$$

$$= F(0, T-1, T) + \ln \frac{u(T-t)}{u(T)} - D(t) \ln k,$$

where $D(t)$ is the number of downward steps of bond prices till time t, i.e. $D(t) = \sum_{s=1}^{t} I(s)$ and

$$I(s) = \begin{cases} 1 & \text{if the bond prices step down at time } s, \\ 0 & \textit{otherwise}. \end{cases}$$

Short rate dynamics in binomial Ho-Lee market

Corollary. In particular, for the risk-free rate in the arbitrage free binomial Ho-Lee model we have

$$r(t) = F(t,t,t+1) = F(0,t,t+1) + \ln \frac{u(1)}{u(t+1)} - D(t)\ln k.$$

Note that clearly $t = D(t) + U(t)$ where $U(t)$ denotes the number of upward steps up to time t.
For r we obtained a random walk type of process with constant volatility (see the term $D(t)\ln k$), however the drift depends on time.

Assumptions of the model

In this subsection we consider a binomial model which is introduced through the spot rate process r, which evolves in a binomial tree according to

$$\mathbb{P}(r(t+1) = r_{i-1} \mid r(t) = r_i) = 1 - \mathbb{P}(r(t+1) = r_{i+1} \mid r(t) = r_i),$$

where r_i is a real constant for all $i \in \mathbb{Z}$, $t \in \mathcal{T}$.
Note that for a finite time horizon T^* we need only finitely many r_i's.

Assumptions of the model: the EMM

Let $\mathbb{P}^*$ be a probability measure which is defined by the conditional probabilities $q_i \in (0,1)$, $i \in \mathbb{Z}$, such that for all $i \in \mathbb{Z}$, $t \in cT$ we have

$$q_i = \mathbb{P}^*(r(t+1) = r_{i+1} \mid r(t) = r_i),$$

and therefore

$$1 - q_i = \mathbb{P}^*(r(t+1) = r_{i-1} \mid r(t) = r_i)$$

Assume, furthermore, that

$$\mathbb{E}_{\mathbb{P}^*}(P(t, t+2) \mid r(t) = r_i) = \mathrm{e}^{-r_i} \left(\mathrm{e}^{-r_{i+1}} q_i + \mathrm{e}^{-r_{i-1}} (1 - q_i)\right).$$

No-arbitrage

Theorem. Under the above assumption $\mathbb{P}^*$ is an EMM in the market and therefore the for the bond prices we have

$$P(t,T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\sum_{s=t}^{T-1} r(s)} \mid \mathcal{F}_t\right)$$
$$= P(t,t+1)\mathbb{E}_{\mathbb{P}^*}\left(P(t+1,T) \mid \mathcal{F}_t\right),$$

$t, T \in \mathcal{T}$, $0 \le t \le T \le T^*$.

A random walk case

Example. Let us assume that there is a $q \in (0,1)$ with $q = q_i$ for all $i \in \mathbb{Z}$, and let $r_i := r_0 + i\delta$, with a positive constant δ.
In this case the spot rate may change by either δ or $-\delta$ in every step and the corresponding probabilities are constant in time.
Hence we obtain a simple random walk model.

Derivative prices

Theorem. Consider the model introduced in this subsection. Let $f(P(T,S))$ is the payoff of a European type of option with maturity S written on a T-bond, where $0 \leq T < S \leq T^*$. Then the unique arbitrage-free price of the option at time t is

$$V(t) = \mathbb{E}_{\mathbb{P}^*}\left(\mathrm{e}^{-\sum_{s=t}^{T-1} r(s)} f(P(T,S)) \mid \mathcal{F}_t\right),$$

$t \in \mathcal{T}$, $0 \leq t \leq T$.

Forward vs futures prices

- Take a T-bond and let $0 \leq t \leq S < T < T^*$.
- Consider a futures contract with maturity S written on the T-bond.
- Denote the futures price at time t of the T-bond corresponding to maturity S by $f(t, S, T)$.
- Similarly, consider a forward contract with maturity S written on the T-bond.
- Denote the forward price at time t of the T-bond corresponding to maturity S by $\tilde{f}(t, S, T)$.

The forward contract

The value of the forward contract is zero when it is issued, and the only cash-flow of the contract is at maturity, which is

$$P(S,T) - K,$$

where K is the delivery (or exercise) price, which is chosen to be the forward price when the contract is written, i.e. $K = \tilde{f}(t,S,T)$, if the contract is issued at time t.

Forward price of the bond

Theorem. For the forward price we have

$$\mathbb{E}_{\mathbb{P}^*}\left(\frac{B(t)}{B(S)}(P(S,T)-\tilde{f}(t,S,T)) \mid \mathcal{F}_t\right)=0,$$

(since such a contract has value zero at the start) which gives

$$\tilde{f}(t,S,T)=\frac{P(t,T)}{P(t,S)},$$

$0 \leq t \leq S < T < T^*$.

The futures contract

The value of the futures contract is zero when it is issued, say, at t, and it is settled at each trading time, hence the cash-flow received by the long position at t is the change of the futures price, i.e. at time n it is

$$f(n, S, T) - f(n-1, S, T), \quad t \leq n \leq S,$$

and the delivery (or exercise) price K of the contract is set to be the forward price when the contract is written, i.e. $K = f(t, S, T)$, if the contract is issued at time t.

Futures price of the bond

Theorem. For the futures prices we have

$$\mathbb{E}_{\mathbb{P}^*}\left(\sum_{n=t+1}^{S}\frac{B(t)}{B(n)}(f(n,S,T)-f(n-1,S,T)) \mid \mathcal{F}_t\right)=0,$$

for all $t\in\mathcal{T}$, $t<S$, which gives

$$f(t,S,T)=\mathbb{E}_{\mathbb{P}^*}(f(t+1,S,T)\mid\mathcal{F}_t)=\mathbb{E}_{\mathbb{P}^*}(P(S,T)\mid\mathcal{F}_t),$$

$0\le t\le S<T<T^*$.

Comparison of forward and futures prices

Remark.
Note that the forward and futures prices on the same bond with the same maturity do not coincide necessarily.
This is due to the random interest rate and the fact that the price of the underlying product (the bond) and the discount rate are not independent.

Basic assumptions

- Given a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_k\}_{k \in \mathbb{Z}_+}$, assume that S_k is an adapted stochastic process, $k \in \mathcal{T}$.
- Notation: $\Delta S_k := S_{k+1} - S_k$.

Interest rates

- We shall write $f_{k,j} := F(k, k+j, k+j+1)$, that is
- $f_{k,j}$ is the **forward interest rate** at time k, with maturity $k+j$, corresponding to the period $[k+j \ , \ k+j+1]$,
- j is time **to** maturity,
- r_k is the spot interest rate for the period $[k, k+1)$,

$$r(k) = f_{k,0} \qquad \forall \quad k \in \mathcal{T}.$$

Forward rate dynamics

$$f_{k+1,j} = f_{k,j} + \alpha_{k,j} + \beta_{k,j}(S_{k+1} - S_k),$$

where $\sigma(\alpha_{k,j}), \sigma(\beta_{k,j}) \subset \mathcal{F}_k$, $k \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, or equivalently

$$f_{k,j} = f_{0,j} + \sum_{i=0}^{k-1} \alpha_{i,j} + \sum_{i=0}^{k-1} \beta_{i,j} \Delta S_i.$$

Zero coupon bond prices

Using continuous compounding convention,
$P(k,\ell)$: price of the zero coupon bond with maturity ℓ at time k,
$P(k,k) := 1$ (no default) and for $0 \leq k \leq \ell \leq T^*$ we have

$$P(k,\ell+1) := P(k,\ell)e^{-F(k,\ell,\ell+1)} = P(k,\ell)e^{-f_{k,\ell-k}},$$

i.e.

$$P(k,\ell) = e^{-\sum_{j=k}^{\ell-1} F(k,j,j+1)} = e^{-\sum_{j=0}^{\ell-k-1} f_{k,j}}.$$

Discounted bond prices

discounting to time $t = 0$:

$$\frac{1}{\prod_{j=0}^{k-1} \mathbf{e}^{r_j}} P(k, \ell) = \mathrm{e}^{-\sum_{j=0}^{k-1} r_j} P_{k,\ell},$$

hence the discount factor is

$$D_k = \mathrm{e}^{-\sum_{j=0}^{k-1} r(j)},$$

i.e.

$$\frac{D_{k+1}}{D_k} = \mathrm{e}^{-r(k)}.$$

Stochastic market discount factor, market price of risk

We assume the existence of a stochastic market discount factor M, which is a key process for pricing assets in the market.
Assume that $\mathbb{E}(\exp\{\phi_k \Delta S_k\}) < \infty$ for $k \in \mathcal{T}$, where ϕ_k is $\mathcal{F}_k$-measurable r.v. Now define $M_0 := 1$ and

$$M_{k+1} := M_k \frac{\mathrm{e}^{-r(k)+\phi_k \Delta S_k}}{\mathbb{E}\left(\mathrm{e}^{\,\phi_k \Delta S_k} \,\middle|\, \mathcal{F}_k\right)},$$

where ϕ_k's are called the **market price of risk** factors.

Absence of arbitrage

Theorem. Consider ('the pricing kernel' or density) $\frac{d\mathbb{P}^*_K}{d\mathbb{P}_K} = \Lambda_K$, where $\mathbb{P}_K = \mathbb{P}|_{\mathcal{F}_K}$, $K \in \mathcal{T}$, $\Lambda_0 := 1$, and

$$\Lambda_{K+1} := \frac{e^{\sum_{k=0}^{K} \phi_k \Delta S_k}}{\prod_{k=0}^{K} \mathbb{E}\left(e^{\phi_k \Delta S_k} \middle| \mathcal{F}_k\right)}.$$

Then the measure $\{\mathbb{P}^*_k\}_{k \in \mathbb{Z}_+}$ are compatible, i.e. $\mathbb{P}^*_{K_1}(A) = \mathbb{P}^*_{K_2}(A)$ for all $0 \leq K_1 \leq K_2 \leq \mathcal{T}$, $A \in \mathcal{F}_{K_1}$ and $\mathbb{P}^* = \mathbb{P}^*_{T^*}$ is equivalent to $\mathbb{P}$.

Furthermore, $M_k P_{k,\ell}$ is $\mathbb{P}$-martingale $\forall \ell$ if and only if there is no arbitrage in the market.

Assumption. Hence, in what follows we shall assume that $M_k P_{k,\ell}$ is a $\mathbb{P}$-martingale $\forall \ell$.

General drift condition

Theorem. Let G_k denote the conditional moment generating function of ΔS_k with respect to $\mathcal{F}_k$ under $\mathbb{P}$. (Clearly, G_k is the regular moment generating function if ΔS_k and $\mathcal{F}_k$ are independent.) The market introduced above is free of arbitrage if and only if for all $0 \leq k < \ell$ we have

$$G_k\left(\phi_k - \sum_{j=0}^{\ell-k-2} \beta_{k,j}\right) = G_k\left(\phi_k\right) \mathrm{e}^{r_k - f_{k,\ell-k-1} + \sum_{j=0}^{\ell-k-2} \alpha_{k,j}},$$

a.s. for all $0 \leq k < \ell \leq \mathcal{T}$.

Drift condition for normal case

Theorem. Assume for all $k \in \mathcal{T}$, $k < T^*$, that ΔS_k is $\mathbb{P}$-independent of $\mathcal{F}_k$ and they have standard normal distribution under $\mathbb{P}$. Then the no-arbitrage condition of the market can be written in the form

$$f_{k,m} = r(k) + \sum_{i=0}^{m-1} \alpha_{k,j} - \frac{1}{2} \left(\sum_{j=0}^{m-1} \beta_{k,j} \right)^2 + \phi_k \sum_{j=0}^{m-1} \beta_{k,j},$$

a.s. for all possible k, m.

Drift condition for normal case (cont.)

Theorem (cont.) Furthermore,

$$f_{k,m} = f_{0,m+k} + \sum_{i=0}^{k-1} a_{i,m+k-i-1} + \sum_{i=0}^{k-1} \beta_{i,m+k-i-1} \Delta S_i,$$

a.s., where

$$a_{i,\ell} := \beta_{i,\ell} \left[\sum_{j=0}^{\ell-1} \beta_{i,j} - \phi_i + \frac{1}{2} \beta_{i,\ell} \right].$$

(Notice that the original drift variables $(\alpha_{i,j})$ are not in the equations.)

Forward rate driven by random fields

One can generalise the above discussed forward rate model by replacing the process S by a random field, that is assuming, that we are given a certain set of random variables $S_{i,j}$, $i, j \in \mathcal{T}$, such that $S_{i,j}$ is $\mathcal{F}_i$-measurable and

$$f_{k+1,j} = f_{k,j} + \alpha_{k,j} + \beta_{k,j}(S_{k+1,j} - S_{k,j}).$$

This way one can obtain a fairly general family of HJM type forward rate models which can include the important properties of the interest rates and assets observed in the market (see Gáll, Pap & Zuijlen (2006)).

Bibliographic notes

For the discussion of the discrete time models in the course we mainly used Chapter 3 in Cairns (2004) and some parts of the paper Gáll, Pap & Zuijlen (2006). Most of the books in the literature of interest rates focus on continuous time models, however we mention the work of Jarrow (1996), which gives an introduction to discrete time interest rate models. We also refer to Hull (2012), where some fundamental financial background and some basic interest rate trees are discussed, in particular in Chapters 30, 31.

References

CAIRNS, A. J. G. (2004), *Interest Rate Models - An introduction*, Princeton University Press, Princeton and Oxford.

GÁLL, J., PAP, G. and ZUIJLEN, M. V. (2006), *Forward interest rate curves in discrete time settings driven by random fields*, Computers & Mathematics with Applications, **51(3-4)**, 387–396.

HULL, J. C (2012), *Options, Futures, and Other Derivatives*, Eighth Edition, Pearson Education Limited (Global Edition.).

JARROW, R. A. (1996), *Modeling Fixed Income Securities and Interest Rate Options*, The McGraw-Hill Companies, Inc., New York.