Tools of stochastic calculus

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part III

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Notations, basic assumptions

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$.

Definition. An adapted process W with $W_0 = 0$ is a standard Brownian motion (or Wiener process) if

- it has continuous sample paths a.s.,
- the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $s \leq t \leq T$,
- the increment $W_t - W_s$ is normally distributed with mean 0 and variance for all $t - s$, $s \leq t \leq T$.

- $L^2(\Omega, \mathcal{F}, \mathbb{P})$: the space of square integrable $\mathcal{F}_T$-measurable r.v.'s.
- A process γ is *progressively measurable*, if the map $(\omega, t) \to \gamma_t(\omega)$ is measurable w.r.t. the σ-algebra $\mathcal{B}([0, t]) \times \mathcal{F}_t$.
- $\mathcal{L}^2(W)$: the class of prog. measurable processes γ for which

$$\mathbb{E}\left(\int_0^T \gamma_u^2 \, du\right) < \infty.$$

- Notation: $||\gamma||^2 := \mathbb{E}\left(\int_0^T \gamma_u^2 \, du\right)$.
- A process γ is said to be *elementary* if it is of the form

$$\gamma_t = \xi_0 I_0 + \sum_{j=0}^{m-1} \xi_j I_{(t_j, t_{j+1}]}(t), \quad t \in [0, T],$$

where ξ_i is $\mathcal{F}_{t_i}$-measurable, $i = 0, \ldots, m-1$, and $0 = t_0 < t_1 < \ldots < t_m = T$.

Itô integral of an elementary process

$$\hat{I}(\gamma) = \int_0^T \gamma_u \, dW_u := \sum_{j=0}^{m-1} \xi_j (W_{t_{j+1}} - W_{t_j}),$$

similarly the integral over $[0, t]$, $t \in [0, T]$, is

$$\hat{I}_t(\gamma) = \int_0^t \gamma_u \, dW_u := \hat{I}(\gamma \mathbf{1}_{[0,t]}) = \sum_{j=0}^{m-1} \xi_j (W_{t_{j+1} \wedge t} - W_{t_j}),$$

Remark: $\hat{I}_t(\gamma)$ is a cont. martingale.

Extension of $\hat{I}$

Lemma. For any elementary process γ

$$||\hat{I}(\gamma)||_{L^2}^2 = \mathbb{E}\left(\int_0^T \gamma_u \, dW_u\right)^2 = ||\gamma||_W^2,$$

i.e. $\hat{I}$ is an isometry. The class of elementary processes is a dense linear subspace of $\mathcal{L}^2(W)$.
$\mathcal{L}^2(W)$ is a Banach space equipped with the norm $||\cdot||_W$.
Hence, there is an extension of $\hat{I}$ such that it is an L^2 isometry.
Definition. The isometry I gives the Itô integral with respect to W, i.e.

$$I_t(\gamma) = \int_0^t \gamma_u \, dW_u := I(\gamma \mathbf{1}_{[0,t]}).$$

Remark. One can extend the definition for set of prog. measurable processes γ with $\mathbb{P}(\int_0^T \gamma_u^2\, du < \infty) = 1$.

Theorem. If the process γ is in $\mathcal{L}^2(W)$ then the Itô integral $I_t(\gamma)$ is a square-integrable continuous martingale.

Itô processes

Definition. A stoc. process X is an Itô process if it is of the form

$$X_t = X_0 + \int_0^t \alpha_u \, du + \int_0^t \beta_u \, dW_u, \quad t \in [0, T],$$

provided that the above integrals are well defined, and α and β are adapted processes.

Another notation for the above integral equation:

$$dX_t = \alpha_t \, dt + \beta_t \, dW_t.$$

Multidimensional Itô integral

Definition.

- d-dimensional standard Brownian motion: $W = (W_1, W_2, \ldots, W_d)$, where W_i's are mutually independent std. Brownian motions, $i = 1, 2, \ldots, d$.
- Let $\gamma = (\gamma^1, \ldots, \gamma^d)$ be an adapted $\mathbb{R}_d$-valued process with $\mathbb{P}(\int_0^T |\gamma|_u^2 \, du < \infty) = 1$, where $|\cdot|$ denotes the Euclidean norm. Then the Itô integral of γ w.r.t. W is

$$I_t(\gamma) = \int_0^t \gamma_u \, dW_u = \sum_{i=1}^d \int_0^t \gamma_u^i \, dW_u^i, \quad t \in [0, T].$$

Itô formula in one dimension

Theorem. Let X be an Itô process,

$$dX_t = \alpha_t \, dt + \beta_t \, dW_t$$

and $g : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a function in $\mathcal{C}^{2,1}(\mathbb{R} \times [0, T], \mathbb{R})$. Then the process $Y_t := g(X_t, t)$ is an Itô process, which has the form:

$$dY_t = \left(g_t(X_t, t) + g_x(X_t, t)\alpha_t + \frac{1}{2} g_{xx}(X_t, t)\beta_t^2 \right) \, dt + g_x(X_t, t)\beta_t \, dW_t.$$

Itô formula, multidimensional case

Theorem. Let X be a k-dimensional Itô process,

$$dX_t^i = \alpha_t^i \, dt + \beta_t^i \, dW_t,$$

where α_i is real valued adapted process, β_i is $\mathbb{R}^d$ valued process s.t. the above integrals exist. Let $g : \mathbb{R}^k \longrightarrow \mathbb{R}$ is a function in $\mathcal{C}^2(\mathbb{R}^k, \mathbb{R})$.
Then the process $Y_t := g(X_t)$ is an Itô process, which has the form:

$$dY_t = \left(\sum_{i=1}^k g_{x_i}(X_t)\alpha_t^i + \frac{1}{2} \sum_{i,j=1}^k g_{x_i x_j}(X_t)\beta_{x_i}\beta_{x_j} \right) \, dt$$

$$+ \sum_{i=1}^k g_{x_i}(X_t)\beta_t^i \, dW_t.$$

Product rule

Corollary. Let X^1 and X^2 be Itô processes,

$$dX_t^i = \alpha_t^i \, dt + \beta_t^i \, dW_t, \quad i = 1, 2.$$

Then $Y_t = X_t^1 X_t^2$ is an Itô process with

$$dY_t = X^1 dX^2 + X^2 dX^1 + \beta_t^1 \beta_t^2 \, dt$$

Remark. In fact $\beta_t^1 \beta_t^2 \, dt$ is the so-called quadratic covariation process $\langle X^1, X^2 \rangle_t$ of X^1 and X^2.

Filtration generated by W

Let W be a (d-dim.) std. Brownian motion.

- Consider first the filtration $(\mathcal{F}_t)_{t=0}^{T^*}$ generated by W, i.e. $\mathcal{F}_t = \sigma\{W_s \mid 0 \leq s \leq t\}$.
- Define $\bar{\mathcal{F}}_t = \sigma\{\mathcal{F}_t \cup \mathcal{N}\}$, where $\mathcal{N}$ is the set of all measurable sets in $\mathcal{F}_{T^*}$ having probability zero.
- Finally, define $\bar{\bar{\bar{\mathcal{F}}}}_t = \cap_{\delta>0}\mathcal{F}_t \bar{+} \delta$.
- The filtration $(\bar{\bar{\bar{\mathcal{F}}}}_t)_{t=0}^{T^*}$ is called the $\mathbb{P}$-complete right continuous version of $(\mathcal{F}_t)_{t=0}^{T^*}$, or simply the $\mathbb{P}$-augmented version of the filtration generated by W. In what follows, if not stated otherwise, given a Brownian motion W we will assume that the filtration is the $\mathbb{P}$-augmented version of the filtration generated by W.

Doléans exponential

Let W be a d-dim. std. Brownian motion. Suppose that γ is an adapted $\mathbb{R}^d$-valued process in $\mathcal{L}(W)$. Then the Doléans exponential of $I_t(\gamma) = \int_0^t \gamma_u \, dW_u$, $t \in [0, T^*]$, is

$$\xi_t = \varepsilon_t \left(\int_0^{\cdot} \gamma_u \, dW_u \right) := \exp \left\{ \int_0^t \gamma_u \, dW_u - \frac{1}{2} \int_0^t |\gamma_u|^2 \, du \right\}.$$

Remark. By Itô formula one can check that ξ_t is the solution of the following SDE:

$$d\xi_t = \xi_t \gamma_t \, dW_t.$$

Equivalent measures

Let $\mathbb{P}^*$ be an equivalent probability measure to $\mathbb{P}$. Denote the Radon-Nikodým derivative (or density) by

$$\eta_{T^*} = \frac{d\mathbb{P}^*}{d\mathbb{P}}$$

and in general the density process by

$$\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_{T^*} \mid \mathcal{F}_t), \quad t \in [0, T^*].$$

- Then the process η is strictly positive (a.s.) and it follows a martingale.
- By the abstract Bayes's formula we have that a process X is a $\mathbb{P}^*$-martingale if and only if the process $X \cdot \eta$ is a $\mathbb{P}$-martingale.

Girsanov's theorem

Theorem. Given a d-dim. std. Brownian motion W and a process γ which is adapted $\mathbb{R}^d$-valued in $\mathcal{L}(W)$ with

$$\mathbb{E}_{\mathbb{P}} = \varepsilon_{T^*}\left(\int_0^\cdot \gamma_u \, dW_u\right) = 1,$$

define the equivalent probability measure $\mathbb{P}^*$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \varepsilon_{T^*}\left(\int_0^\cdot \gamma_u \, dW_u\right), \text{ a.s.}$$

Then the process W^* defined by

$$W_t^* := W_t - \int_0^t \gamma_u \, du, \quad t \in [0, T^*],$$

is a standard d-dimensional Brownian motion w.r.t. $\mathbb{P}^*$.

Remark. Given an Itô process under $\mathbb{P}$ of the form

$$dX_t = \alpha(X_t, t)\, dt + \beta(X_t, t)\, dW_t$$

(with well defined integrals) suppose that $\alpha^*(X_t, t) = \alpha(X_t, t) + \beta(X_t, t) \cdot \gamma_t$ a.s., $t \in [0, T^*]$. Then we have under $\mathbb{P}^*$

$$dX_t = \alpha^*(X_t, t)\, dt + \beta(X_t, t)\, dW_t^*.$$

Theorem. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbb{P}^*$ be an equivalent probability measure to $\mathbb{P}$ with $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \eta$ a.s. Suppose that $\mathcal{G}$ is a σ-algebra, $\mathcal{G} \subset \mathcal{F}$, and ξ is an integrable random variable w.r.t. $\mathbb{P}^*$. Then

$$\mathbb{E}_{\mathbb{P}^*}(\xi \mid \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(\xi\eta \mid \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{G})} \text{ a.s.}$$

Proof. Let $\bar{\xi} = \xi\eta$, then one can easily show that $\mathbb{E}_{\mathbb{P}}(\bar{\xi}|\mathcal{G}) = \mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{G})\mathbb{E}_{\mathbb{P}}(\eta|\mathcal{G})$, which gives the statement.

Bibliographic notes

The main tools and facts of stochastic integration are summarised in the main books that we use for this course, hence we refer here to Appendix A in Cairns (2004), Appendix C in Brigo & Mercurio (2006), and Appendix B in Musiela & Rutkowski (2005). We mostly used the structure and notations of the latter one. For a more detailed discussion on stochastic integration one can use several books, we refer to Chung & Williams (1990), which gives a nice introduction to the field.

References

Brigo, D. and Mercurio, F. (2006), *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, Springer, Berlin Heidelberg New York.

Cairns, A. J. G. (2004), *Interest Rate Models - An introduction*, Princeton University Press, Princeton and Oxford.

Chung, K. L. and Williams, R. J. (1900), *Introduction to Stochastic Integration*, Second Edition, BirkHäuser, Boston-Basel-Berlin.

Musiela, M. and Rutkowski, M. (2005), *Martingale Methods in Financial Modeling*, second edition, Springer-Verlag, Berlin, Heidelberg.