Chapter 6

The Time Value of Money: Annuities and Other Topics
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  1. Annuities
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  3. Complex Cash Flow Streams
Learning Objectives

1. Distinguish between an ordinary annuity and an annuity due, and calculate present and future values of each.

2. Calculate the present value of a level perpetuity and a growing perpetuity.

3. Calculate the present and future value of complex cash flow streams.
6.1 Annuities
Ordinary Annuities

• An **annuity** is a series of *equal dollar* payments that are made at the end of equidistant points in time such as monthly, quarterly, or annually over a *finite period* of time.

• If payments are made at the end of each period, the annuity is referred to as **ordinary annuity**.
Ordinary Annuities (cont.)

• **Example 6.1** How much money will you accumulate by the end of year 10 if you deposit $3,000 each for the next ten years in a savings account that earns 5% per year?

• We can determine the answer by using the equation for computing the future value of an ordinary annuity.
The Future Value of an Ordinary Annuity

\[ FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

- \( FV_n \) = FV of annuity at the end of \( n \)th period.
- \( PMT \) = annuity payment deposited or received at the end of each period
- \( i \) = interest rate per period
- \( n \) = number of periods for which annuity will last
The Future Value of an Ordinary Annuity (cont.)

\[ FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

- \( FV = \$3000 \left\{ \left[ (1+.05)^{10} - 1 \right] \div (.05) \right\} \)
  
  \[
  = \$3,000 \left\{ [0.63] \div (.05) \right\} \\
  = \$3,000 \left\{ 12.58* \right\} \\
  = \$37,740
  \]

(*use appendix D)
Solving for PMT in an Ordinary Annuity

• Instead of figuring out how much money you will accumulate (i.e. FV), you may like to know how much you need to save each period (i.e. PMT) in order to accumulate a certain amount at the end of n years.

• In this case, we know the values of n, i, and $FV_n$ in equation 6-1c and we need to determine the value of PMT.
Solving for PMT in an Ordinary Annuity (cont.)

• **Example 6.2:**

Suppose you would like to have $25,000 saved 6 years from now to pay towards your down payment on a new house. If you are going to make equal annual end-of-year payments to an investment account that pays 7 per cent, how big do these annual payments need to be?
Solving for PMT in an Ordinary Annuity (cont.)

\[ FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

• Here we know, \( FV_n = $25,000; \) \( n = 6; \) and \( i = 7\% \) and we need to determine PMT.

• \$25,000 = PMT \left\{ \left[ (1+.07)^6 - 1 \right] \div (.07) \right\}
  = PMT\left\{ [.50] \div (.07) \right\}
  = PMT \{7.153\}

\$25,000 \div 7.153 = PMT = \$3,495.03
The Present Value of an Ordinary Annuity

- The present value of an ordinary annuity measures the value today of a stream of cash flows occurring in the future.

- For example, we will compute the PV of ordinary annuity if we wish to answer the question: what is the value today equivalent of receiving every year for the next years if the interest rate is fixed?
The Present Value of an Ordinary Annuity (cont.)

\[
\text{Present Value} = PMT \left[ \frac{1 - \frac{1}{(1 + i)^n}}{i} \right]
\]

- \( PMT = \) annuity payment deposited or received at the end of each period.
- \( i = \) discount rate (or interest rate) on a per period basis.
- \( n = \) number of periods for which the annuity will last.
What is the present value of an annuity of $10,000 to be received at the end of each year for 10 years given a 10 percent discount rate?

- **Using the Mathematical Formula**

  
  \[
  \text{Present Value} = PMT \left[ \frac{1 - \frac{1}{(1 + i)^n}}{i} \right]
  \]

- **PV =** $10,000 \left\{ \left[ 1 - \frac{1}{(1.10)^{10}} \right] \div (0.10) \right\}

  = $10,000 \left\{ [0.6145\text{]*} \div (0.10) \right\}

  = $10,000 \left\{ 6.145 \right\}

  = $61,445
  
  (*use appendix E)
Practice - Amortized Loans

• An amortized loan is a loan paid off in equal payments – consequently, the loan payments are an annuity.

• Examples: Home mortgage loans, Auto loans
Amortized Loans (cont.)

In an amortized loan:

• the *present value* can be thought of as the amount borrowed,

• $n$ is the number of periods the loan lasts for,

• $i$ is the interest rate per period,

• *future value* takes on zero because the loan will be paid off after $n$ periods, and *payment* is the loan payment that is made.
Amortized Loans (cont.)

- **Example 6.5** Suppose you plan to get a $9,000 loan from a furniture dealer at 18% annual interest with annual payments that you will pay off in over five years. What will your annual payments be on this loan?

- \[ 9000 = \text{PMT} \left\{ \frac{1-(1/(1.18)^5)}{(1.18)} \right\} \]
  \[ \text{PMT} = \frac{9000}{3.127} = $2,878 \]
# The Loan Amortization Schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount Owed on Principal at the Beginning of the Year (1)</th>
<th>Annuity Payment (2)</th>
<th>Interest Portion of the Annuity (3) (= (1) \times 18%)</th>
<th>Repayment of the Principal Portion of the Annuity (4) (=(2) - (3))</th>
<th>Outstanding Loan Balance at Year end, After the Annuity Payment (5) (=(1) - (4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$9,000</td>
<td>$2,878</td>
<td>$1,620.00</td>
<td>$1,258.00</td>
<td>$7,742.00</td>
</tr>
<tr>
<td>2</td>
<td>$7,742</td>
<td>$2,878</td>
<td>$1,393.56</td>
<td>$1,484.44</td>
<td>$6,257.56</td>
</tr>
<tr>
<td>3</td>
<td>$6257.56</td>
<td>$2,878</td>
<td>$1,126.36</td>
<td>$1,751.64</td>
<td>$4,505.92</td>
</tr>
<tr>
<td>4</td>
<td>$4,505.92</td>
<td>$2,878</td>
<td>$811.07</td>
<td>$2,066.93</td>
<td>$2,438.98</td>
</tr>
<tr>
<td>5</td>
<td>$2,438.98</td>
<td>$2,878</td>
<td>$439.02</td>
<td>$2,438.98</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
• We can observe the following from the table:
  – Size of each payment remains the same.
  – However, Interest payment declines each year as the amount owed declines and more of the principal is repaid.
Amortized Loans with Monthly Payments (cont.)

- **Mathematical Formula**

\[
PV = PMT \left[ \frac{1 - \frac{1}{(1 + \text{annual interest rate}/m)^{\text{number of years} \times m}}}{\text{annual interest rate}/m} \right]
\]

- Here annual interest rate = .06, number of years = 30, m=12, PV = $300,000

- $300,000 = PMT \left\{ \left[ 1 - (1/(1.06)^{30 \times 12}) \right] \div (.06/12) \right\}$
- $300,000 = PMT [166.79]
- $300,000 \div 166.79 = $1798.67
Annuities Due

- **Annuity due** is an annuity in which all the cash flows occur at the beginning of the period.
- For example, rent payments on apartments are typically annuity due as rent is paid at the beginning of the month.
Annuities Due: Future Value

- Computation of future value of an annuity due requires compounding the cash flows for one additional period, beyond an ordinary annuity.

\[ FV_n(\text{annuity due}) = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] (1 + i) \]
Annuities Due: Present Value

- Since with annuity due, each cash flow is received one year earlier, its present value will be discounted back for one less period.

\[ PV(\text{annuity due}) = PMT \left[ \frac{1 - \frac{1}{(1 + i)^n}}{i} \right] (1 + i) \]
6.2 Perpetuities
Perpetuities

- A **perpetuity** is an annuity that continues forever or has no maturity.

For example, a dividend stream on a share of preferred stock. There are two basic types of perpetuities:

- **Growing perpetuity** in which cash flows grow at a constant rate, $g$, from period to period.
- **Level perpetuity** in which the payments are constant rate from period to period.
Present Value of a Level Perpetuity

\[ PV = \frac{PMT}{i} \]

- PV = the present value of a level perpetuity
- PMT = the constant dollar amount provided by the perpetuity
- i = the interest (or discount) rate per period
Present Value of a Level Perpetuity

- Example 6.6 What is the present value of $600 perpetuity at 7% discount rate?

\[
PV = \frac{PMT}{i}
\]

- \( PV = \$600 \div .07 = \$8,571.43 \)
Present Value of a Growing Perpetuity

• In growing perpetuities, the periodic cash flows grow at a constant rate each period.

• The present value of a growing perpetuity can be calculated using a simple mathematical equation.
Present Value of a Growing Perpetuity (cont.)

\[ PV = \frac{PMT_{\text{period 1}}}{i - g} \]

- PV = Present value of a growing perpetuity
- \( PMT_{\text{period 1}} \) = Payment made at the end of first period
- \( i \) = rate of interest used to discount the growing perpetuity’s cash flows
- \( g \) = the rate of growth in the payment of cash flows from period to period
- \( i > g \)
6.3 Complex Cash Flow Streams
Complex Cash Flow Streams

- The cash flows streams in the business world may not always involve one type of cash flows.
- The cash flows may have a mixed pattern. For example, different cash flow amounts mixed in with annuities.
Check Yourself

What is the present value of cash flows
• of $300 at the end of years 1 through 5,
• a cash flow of negative $600 at the end of year 6,
• and cash flows of $800 at the end of years 7-10
• if the appropriate discount rate is 10%?
Step 1: Picture the Problem

i=10%
Years
Cash flows

0 1-5 6 7-10
$300  -$600  $800

PV equals the PV of ordinary annuity
PV equals PV of $600 discounted back 6 years
PV in 2 steps: (1) PV of ordinary annuity for 4 years (2) PV of step 1 discounted back 6 years
Step 2: Decide on a Solution Strategy (cont.)

- The $800 annuity will have to be solved in two stages:
  - Determine the present value of ordinary annuity for four years.
  - Discount the single cash flow (obtained from the previous step) back 6 years to the present using equation 5-2.
Step 3: Solve

- Using the Mathematical Formula
- (Step 1) PV of $300 ordinary annuity

\[
\text{Present Value} = PMT \left[ 1 - \frac{1}{(1 + i)^n} \right] \div i
\]

\[
\text{PV} = 300 \left\{ \left[ 1 - \frac{1}{(1 + .10)^5} \right] \div (.10) \right\}
\]

\[
= 300 \left\{ [0.379] \div (.10) \right\} = 300 \times 3.79
\]

\[
= 1,137.24
\]
Step 3: Solve (cont.)

- Step (2)
- PV of -$600 at the end of year 6

\[
PV = \frac{FV}{(1+i)^n}
\]

\[
PV = -\frac{$600}{(1.1)^6} = $338.68
\]
Step 3: Solve (cont.)

- Step (3): PV of $800 in years 7-10
- First, find PV of ordinary annuity of $800 for 4 years.

\[
\text{PV} = 800 \left\{ \frac{1 - (1/(1.10)^4)}{.10} \right\} \\
= 800 \left\{ \frac{.317}{.10} \right\} \\
= 800 \times 3.17 \\
= \$2,535.89
\]
Step 3: Solve (cont.)

- **Second**, find the present value of $2,536 discounted back 6 years at 10%.

\[
PV = \frac{FV}{(1+i)^n}
\]

\[
PV = \frac{2,536}{(1.1)^6}
\]

\[
= $1431.44
\]
Step 3: Solve (cont.)

- Present value of complex cash flow stream
  = sum of step (1), step (2), step (3)
  = $1,137.24 - $338.68 + $1,431.44
  = **$2,229.82**
Step 4: Analyze

• This example illustrates that a complex cash flow stream can be analyzed using the same mathematical formulas.

• If cash flows are brought to the same time period, they can be added or subtracted to find the total value of cash flow at that time period.